

On the theory of neutrino mixing and oscillations¹

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Abstract

A brief review of the status of neutrino oscillations is given. The phenomenology of neutrino mixing and the standard seesaw mechanism of neutrino mass generation is discussed. Different approaches to neutrino oscillations are considered and compared. The role of the Heisenberg space-momentum uncertainty relation and the Mandelstam-Tamm time-energy uncertainty relation in neutrino oscillations is discussed in some detail.

1 Introduction. Present status of neutrino oscillations

The observation of neutrino oscillations in the solar, atmospheric, reactor and accelerator neutrino experiments [1, 2, 3, 4, 5] is one of the most important recent discoveries in particle physics. Small neutrino masses and peculiar neutrino mixing are commonly considered as a signature of new physics beyond the Standard Model.

Existing neutrino oscillation data (with the exception of the LSND [6] and recent MiniBooNE [7] antineutrino data) can be perfectly described if we assume that the number of neutrinos with definite masses ν_i is equal to the number of flavor neutrinos which, as was proved by the LEP experiments, is equal to three.

Neutrino oscillation data are usually analyzed under the assumption that the flavor neutrino transition probability in vacuum is given by the following standard expression (see, for example, [9])

$$P(\nu_l \rightarrow \nu_{l'}) = \left| \sum_i U_{\nu l i} e^{-i \frac{\Delta m_{ki}^2 L}{2E}} U_{li}^* \right|^2 = \left| \sum_{i \neq k} U_{\nu l i} (e^{-i \frac{\Delta m_{ki}^2 L}{2E}} - 1) U_{li}^* + \delta_{ll} \right|^2. \quad (1)$$

¹Lectures given at IVth International Pontecorvo Neutrino Physics School (26.09-06.10, 2010, Alushta, Crimea, Ukraine).

Here L is the source-detector distance, E is the neutrino energy, U is the unitary mixing matrix, $\Delta m_{ik}^2 = m_k^2 - m_i^2$.

In the case of three-neutrino mixing the unitary 3×3 Pontecorvo-MNS mixing matrix [10, 11] is usually parameterized by the three Euler angles θ_{12} , θ_{23} , θ_{13} and one CP phase δ . It has the following form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

and neutrino transition probabilities in vacuum are characterized by six parameters: θ_{12} , θ_{23} , θ_{13} , δ , Δm_{12}^2 and Δm_{23}^2 .

From the analysis of the existing neutrino oscillation data follows, however, that two parameters are small:

$$\frac{\Delta m_{12}^2}{\Delta m_{23}^2} \simeq \frac{1}{30}, \quad \sin^2 \theta_{13} \lesssim 4 \cdot 10^{-2}. \quad (3)$$

If we neglect the contribution of the small parameters (leading approximation) a rather simple picture of neutrino oscillations has emerged (see, [9]). In this approximation neutrino oscillations in the atmospheric (accelerator) region of $\frac{L}{E}$ ($\frac{\Delta m_{23}^2 L}{2E} \gtrsim 1$) are two-neutrino $\nu_\mu \rightleftharpoons \nu_\tau$ ($\bar{\nu}_\mu \rightleftharpoons \bar{\nu}_\tau$) oscillations. The $\nu_\mu \rightarrow \nu_\mu$ ($\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$) survival probability is given in this case by the standard two-neutrino expression

$$P(\nu_\mu \rightarrow \nu_\mu) = P(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu) = 1 - \frac{1}{2} \sin^2 2\theta_{23} (1 - \cos \Delta m_{23}^2 \frac{L}{2E}). \quad (4)$$

Neutrino oscillations in the reactor KamLAND region of $\frac{L}{E}$ ($\frac{\Delta m_{12}^2 L}{2E} \gtrsim 1$) are $\bar{\nu}_e \rightleftharpoons \bar{\nu}_{\mu,\tau}$ oscillations. The $\bar{\nu}_e \rightarrow \bar{\nu}_e$ survival probability is given in the leading approximation by the expression

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2} \sin^2 2\theta_{12} (1 - \cos \Delta m_{12}^2 \frac{L}{2E}). \quad (5)$$

In the leading approximation the probability of the solar neutrinos to survive is given by the two-neutrino ν_e survival probability in matter which depends on Δm_{12}^2 , $\sin^2 \theta_{12}$ and the electron number density.

The leading approximation gives the dominant contribution to the transition probabilities: the values of the parameters Δm_{12}^2 , Δm_{23}^2 , $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$,

which are determined from the two-neutrino and the three-neutrino analysis, are practically the same.

From the three-neutrino analysis of the Super-Kamiokande atmospheric neutrino data ² [2] the following best fit values of the parameters are found

$$\Delta m_A^2 = 2.1 \cdot 10^{-3} \text{ eV}^2, \quad \sin^2 \theta_{23} = 0.5, \quad \sin^2 \theta_{13} = 0.0. \quad (6)$$

In the case of the normal (inverted) neutrino mass spectrum the following 90% CL limits were inferred

$$1.9 (1.7) \cdot 10^{-3} \leq \Delta m_A^2 \leq 2.6 (2.7) \cdot 10^{-3} \text{ eV}^2, \quad 0.407 \leq \sin^2 \theta_{23} \leq 0.583. \quad (7)$$

For the parameter $\sin^2 \theta_{13}$ the following bounds were obtained

$$\sin^2 \theta_{13} \leq 4 \cdot 10^{-2} (9 \cdot 10^{-2}). \quad (8)$$

The Super-Kamiokande evidence for neutrino oscillations was confirmed by the accelerator long-baseline K2K [4] and MINOS [5] experiments. From the two-neutrino analysis of the MINOS data was found

$$\Delta m_A^2 = (2.43 \pm 0.13) \cdot 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{23} > 0.90. \quad (9)$$

From the three-neutrino global analysis of the KamLAND reactor and solar data was obtained [3]

$$\Delta m_S^2 = (7.50^{+0.19}_{-0.20}) \cdot 10^{-5} \text{ eV}^2, \quad \tan^2 \theta_{12} = 0.452^{+0.035}_{-0.032} \quad (10)$$

For the parameter $\sin^2 \theta_{13}$ was found

$$\sin^2 \theta_{13} = 0.020^{+0.016}_{-0.018} \quad (11)$$

²In the case of the three-neutrino analysis of the neutrino oscillation data it is important to take into account that neutrino masses are labeled differently for the normal neutrino mass spectrum (NS) ($m_1 < m_2 < m_3$; $\Delta m_{12}^2 \ll \Delta m_{23}^2$) and for the inverted mass spectrum (IS) ($m_3 < m_1 < m_2$; $\Delta m_{12}^2 \ll |\Delta m_{13}^2|$). The smaller and larger neutrino mass squared differences (the same for both neutrino mass spectra) are equal in NS (IS) Δm_{12}^2 (Δm_{12}^2) and Δm_{23}^2 ($|\Delta m_{13}^2|$), respectively. Thus, we can not use the Δm_{ik}^2 notation in the case of the three-neutrino analysis of the data. One of the possibility is to use for the larger and smaller neutrino mass-squared differences, independently of the character of the neutrino mass spectrum, the notations Δm_A^2 and Δm_S^2 . Notice that for both neutrino mass spectra the elements of the neutrino mixing matrix U_{li} are usually parameterized *in the same way* (inspite that they have different meaning).

Finally, from the short baseline reactor experiment CHOOZ [12] the following upper bound was obtained for the parameter $\sin^2 \theta_{13}$

$$\sin^2 \theta_{13} < 4 \cdot 10^{-2}. \quad (12)$$

Let us also notice that from the tritium experiments Mainz [13] and Troitsk [14] the following upper bounds for the absolute value of neutrino mass were found

$$m_\beta \leq 2.3 \text{ eV (Mainz)} \quad m_\beta \leq 2.2 \text{ eV (Troitsk)} \quad (13)$$

2 QFT basics of neutrino oscillations

Our understanding of neutrino oscillations is based on the following assumptions:

I. The Lagrangian of the electroweak interaction is the Standard Model charged current and neutral current Lagrangians. The leptonic part of the CC Lagrangian is given by the following expression

$$\mathcal{L}_{\mathcal{I}}^{cc}(x) = -\frac{g}{\sqrt{2}} \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x) \gamma_\alpha l_L(x) W^\alpha(x) + \text{h.c.} \quad (14)$$

Here g is the electroweak constant, $l_L(x) = (\frac{1-\gamma_5}{2}) l(x)$ is the left-handed component of the leptonic field $l(x)$ and $W^\alpha(x)$ is the field of the vector W^\pm -bosons.

II. The flavor (active) fields $\nu_{lL}(x)$ in the Lagrangian (14) are mixtures of the fields of neutrinos with definite masses

$$\nu_{lL}(x) = \sum_i U_{li} \nu_{iL}(x). \quad (15)$$

Here $\nu_i(x)$ is the field of neutrinos with mass m_i and U is a unitary mixing matrix.

The interaction (14) follows from the requirements of the local $SU(2) \times U(1)$ invariance. It was confirmed with high accuracy by numerous experiments on the study of the weak interaction processes.

The existence of the neutrino mixing is confirmed by the neutrino oscillation experiments. Four neutrino oscillation parameters are known with accuracies in the range (3-10)%. However, there are many unknowns in the mixing relation (15). We do not know

- Are neutrinos with definite masses Majorana or Dirac particles?
- Is the number of the neutrinos with definite masses equal to the number of flavor neutrinos (three) or larger (in this case sterile neutrinos must exist)?
- What is the value of the parameter $\sin^2 \theta_{13}$?
- What is the value of the CP phase δ ?
- What is the character of the neutrino mass spectrum (normal or inverted)?
- etc.

We believe that the resolution of these problems apparently will allow to solve the most important problem: *What is the origin of small neutrino masses and neutrino mixing?*

Neutrino masses and neutrino mixing are due to *the neutrino mass term of the Lagrangian*. According to the Standard Model mass terms of quarks and leptons are generated by the spontaneous violation of the electroweak symmetry. The origin of the neutrino mass term at present is unknown. We will consider in this section a general theoretical framework for possible neutrino mass terms. In the next section we will discuss the most popular seesaw mechanism of neutrino mass generation.

Any mass term is a sum of Lorentz-invariant products of the left-handed and right-handed components of a field. Three left-handed flavor neutrino fields $\nu_{lL}(x)$ must enter into the mass term. Do we need other fields to build the mass term? Generally not, if we assume that the lepton number is not conserved. This was shown for the first time in [15].

In fact, it is easy to show that $(\nu_{lL}(x))^c = C\bar{\nu}_{lL}(x)^T$ is the right-handed component ($C\gamma_\alpha^T C^{-1} = -\gamma_\alpha$, $C^T = -C$).³ From $\nu_{lL}(x)$ and $(\nu_{lL}(x))^c$ we can build the following **Majorana mass term**

$$\mathcal{L}^M = -\frac{1}{2} \bar{n}_L M^L (n_L)^c + \text{h.c.} \quad (16)$$

³The left-handed component satisfies the condition $\gamma_5 \nu_L = -\nu_L$. From this relation we have $\gamma_5^T \bar{\nu}_L^T = \bar{\nu}_L^T$. Taking into account that $C\gamma_5^T C^{-1} = \gamma_5$ we find $\gamma_5(\nu_L)^c = (\nu_L)^c$. This relation means that $(\nu_L)^c$ is the right-handed component.

Here

$$n_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \quad (17)$$

and M^L is a 3×3 complex nondiagonal matrix. Taking into account the Fermi-Dirac statistics of the neutrino field we have

$$\bar{n}_L M^L (n_L)^c = \bar{n}_L M^L C \bar{n}_L^T = -\bar{n}_L (M^L)^T C^T \bar{n}_L^T = \bar{n}_L (M^L)^T (n_L)^c. \quad (18)$$

Thus, M^L must be a symmetrical matrix. A symmetrical, complex matrix can be diagonalized with the help of one unitary matrix:

$$M^L = U m U^T, \quad (19)$$

$U^\dagger U = 1$ and m is a diagonal matrix ($m_{ik} = m_i \delta_{ik}$, $m_i > 0$). From (16) and (19) we find

$$\mathcal{L}^M = -\frac{1}{2} \bar{\nu} m \nu = -\frac{1}{2} \sum_{i=1}^3 m_i \bar{\nu}_i \nu_i. \quad (20)$$

Here

$$\nu = U^\dagger n_L + (U^\dagger n_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (21)$$

From (20) and (21) we conclude the following:

1. $\nu_i(x)$ is the neutrino field with the mass m_i
2. The field $\nu_i(x)$ satisfies the Majorana condition

$$\nu_i^c(x) = \nu_i(x). \quad (22)$$

From this condition follows that

$$\nu_i(x) = \int \frac{1}{(2\pi)^{3/2} \sqrt{2p_0}} (a_{ir}(p) u^r(p) e^{-ipx} + a_{ir}^\dagger(p) u^r(-p) e^{ipx}) d^3p. \quad (23)$$

Here $a_{ir}(p)$ and $a_{ir}^\dagger(p)$ are the operators of absorption and creation, respectively, of a neutrino with mass m_i , momentum p , and helicity r .

Thus, if the neutrino field satisfies the Majorana condition (22) there is no notion of antineutrino (or, in another words, neutrino and antineutrino are identical). This is connected with the fact that the mass term (16) is not invariant under the global gauge transformation $\nu_{lL} \rightarrow e^{i\Lambda} \nu_{lL}$ i.e., there is no conserved lepton number which would allow to distinguish neutrino and antineutrino. Notice that the Majorana mass term (16) can not be generated in the framework of the SM with a Higgs doublet (Higgs triplets are necessary).⁴

3. From (21) follows that the flavor field $\nu_{lL}(x)$ is a mixture of three Majorana fields $\nu_{iL}(x)$:

$$\nu_{lL}(x) = \sum_i U_{li} \nu_{iL}(x). \quad (24)$$

We will assume now that not only flavor fields $\nu_{lL}(x)$, components of the lepton doublets, but also singlet (sterile) fields $\nu_{lR}(x)$ enter into the neutrino mass term. There can be two different mass terms in this case. We will consider first **the Dirac mass term**

$$\mathcal{L}^D(x) = - \sum_{l'l} \bar{\nu}_{l'L}(x) M_{l'l}^D \nu_{lR}(x) + \text{h.c.}, \quad (25)$$

where M^D is a 3×3 complex matrix.

The matrix M^D can be diagonalized by a biunitary transformation. We have

$$M^D = U^\dagger m V, \quad (26)$$

where U and V are unitary matrices and $m_{ik} = m_i \delta_{ik}$. From (25) and (26) we find

$$\mathcal{L}^D(x) = \sum_{i=1}^3 m_i \bar{\nu}_i(x) \nu_i(x), \quad (27)$$

⁴ It is clear from the derivation we presented that the fact that neutrinos with definite masses can be Majorana particles is based on the Fermi-Dirac property of neutrino fields. If we assume that neutrino fields are Bose-Einstein fields (this possibility was discussed in [16]) then neutrinos with definite masses can not be Majorana particles. We can see this considering the mass term for a Majorana particle with a mass m . We have $\mathcal{L}^M = -\frac{1}{2} m \bar{\nu} \nu$, where $\nu = \nu^c = C \bar{\nu}^T$. From this last relation we obtain $\bar{\nu}^c = -\nu^T C^{-1}$. Now we have $\bar{\nu} \nu = \bar{\nu}^c \nu^c = -\nu^T C^{-1} C \bar{\nu}^T = -(\nu^T \bar{\nu}^T)^T = +\bar{\nu} \nu (\text{Fermi}) = -\bar{\nu} \nu (\text{Bose})$. Thus, for a "bosonic neutrino" $\bar{\nu} \nu \equiv 0$.

where

$$\nu_i(x) = \sum_l U_{il}^\dagger \nu_{lL}(x) + \sum_l V_{il} \nu_{lR}(x). \quad (28)$$

From (27) and (28) we can make the following conclusions:

1. The field $\nu_i(x)$ is the field of neutrinos with the mass m_i .
2. The flavor fields $\nu_{lL}(x)$ are connected with the left-handed components of the fields of neutrinos with definite masses by the mixing relation

$$\nu_{lL}(x) = \sum_{i=1}^3 U_{li} \nu_{iL}(x). \quad (29)$$

The Lagrangian with the neutrino mass term (25) is invariant under the global phase transformations

$$\nu_i(x) \rightarrow e^{i\Lambda} \nu_i(x), \quad l(x) \rightarrow e^{i\Lambda} l(x), \quad q(x) \rightarrow q(x), \quad (30)$$

where Λ is an arbitrary constant. From the invariance under the transformations (27) follows that the total lepton number L , the same for e , μ and τ , is conserved. The field $\nu_i(x)$ is the four-component Dirac field of neutrinos and antineutrinos with the same mass m_i and different lepton numbers ($L(\nu_i) = 1, L(\bar{\nu}_i) = -1$).⁵

The Dirac neutrino mass term can be generated by the standard Higgs mechanism, which is responsible for the generation of the masses of quarks and leptons. However, this mechanism can not explain the smallness of the neutrino masses with respect to the masses of quarks and leptons.

There is no fundamental principle which requires the conservation of the lepton number L . The baryon asymmetry of the Universe signifies that the baryon number is violated. It is natural to assume that in some interaction the lepton number is also violated. If this interaction is relevant for the generation of the neutrino masses, the neutrino mass term will violate the lepton number. The most general neutrino mass term which violates the lepton number is **the Dirac and Majorana mass term**

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} \sum_{l'l} \bar{\nu}_{l'L} M_{l'l}^L (\nu_{lL})^c - \sum_{l'l} \bar{\nu}_{l'L} M_{l'l}^D \nu_{lR} - \frac{1}{2} \sum_{l'l} \overline{(\nu_{l'R})^c} M_{l'l}^R \nu_{lR} + \text{h.c.} \quad (31)$$

⁵From (29) and (30) we find $\nu_{lL}(x) \rightarrow e^{i\Lambda} \nu_{lL}(x)$. Thus, for the flavor neutrinos we have: $L(\nu_l) = 1, L(\bar{\nu}_l) = -1$.

Here M^L and M^R are complex, symmetrical 3×3 matrices and M^D is a complex 3×3 matrix. After the diagonalization of this mass term we find

$$\nu_{lL} = \sum_{i=1}^6 U_{li} \nu_{iL}, \quad (\nu_{lR})^c = \sum_{i=1}^6 U_{\bar{l}i} \nu_{iL} \quad l = e, \mu, \tau \quad (32)$$

and

$$\mathcal{L}^{D+M}(x) = -\frac{1}{2} \sum_{i=1}^6 m_i \bar{\nu}_i(x) \nu_i(x). \quad (33)$$

Here the field $\nu_i(x)$ satisfies the condition

$$\nu_i(x) = \nu_i^c(x) = C \bar{\nu}_i^T(x) \quad (34)$$

and U in (32) is a 6×6 unitary mixing matrix. From (33) and (34) follows that the field $\nu_i(x)$ is a field of Majorana particles with mass m_i .

From the consideration of the Dirac and Majorana mass term we can conclude that the number of the massive neutrinos can be larger than the number of the flavor neutrinos (three). Let us write in general

$$\nu_{lL} = \sum_{i=1}^{3+n_s} U_{li} \nu_{iL} \quad l = e, \mu, \tau \quad (35)$$

and

$$\nu_{sL} = \sum_{i=1}^{3+n_s} U_{si} \nu_{iL} \quad s = s_1, \dots, s_{n_s} \quad (36)$$

Thus, we assumed that the three flavor neutrino fields ν_{lL} are mixtures of the left-handed components of $3 + n_s$ massive fields. This means that other n_s mixtures of left-handed components of the same $3 + n_s$ massive fields must exist. We denoted them ν_{sL} . The fields ν_{sL} do not enter into the standard weak interaction Lagrangian and are called sterile fields.

All flavor neutrinos (ν_e, ν_μ, ν_τ) were observed in experiments. Sterile neutrinos ν_s can not be produced in weak processes. There are two ways to reveal the existence of the sterile neutrinos.

I. If neutrinos are detected via the observation of NC processes the sum of the probabilities of the transitions into all flavor neutrinos $\sum_{l'=e,\mu,\tau} P(\nu_l \rightarrow \nu_{l'})$ will be measured. If there are no transitions into sterile neutrinos $\sum_{l'=e,\mu,\tau} P(\nu_l \rightarrow \nu_{l'}) = 1$ and no oscillations will be observed. If there are transitions into

sterile neutrinos $\sum_{l'=e,\mu,\tau} P(\nu_l \rightarrow \nu_{l'}) = 1 - \sum_s P(\nu_l \rightarrow \nu_s)$ and neutrino oscillations can be observed.

II. Neutrino oscillations with two neutrino mass-squared differences Δm_A^2 and Δm_S^2 were observed in different experiments. If oscillations with additional mass-squared difference(s) will be measured this will be the proof of the existence of sterile neutrino(s).

During many years the LSND indication [6] in favor of the $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transition with $\Delta m^2 \simeq 1 \text{ eV}^2$ ($\Delta m^2 \gg \Delta m_{A,S}^2$). In the MiniBooNE experiment this indication was checked. In the channel $\nu_\mu \rightarrow \nu_e$ the LSND result was not confirmed [8]. In the channel $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ some indication in favor of neutrino oscillations, compatible with the LSND result, was obtained [7]. Further experiments are necessary in order to test the idea of a possible existence of the sterile neutrinos.

3 On the seesaw mechanism of neutrino mass generation

Neutrino masses are many orders of magnitude smaller than masses of quarks and leptons. Let us consider, for example, the masses of the particles of the third family. We have

$$m_t \simeq 1.7 \cdot 10^2 \text{ GeV}, \quad m_b \simeq 4.7 \text{ GeV}, \quad \mathbf{m}_3 \leq \mathbf{2.3} \cdot 10^{-9} \text{ GeV}, \quad m_\tau \simeq 1.8 \text{ GeV}. \quad (37)$$

From these values we can conclude that it is very unlikely that the masses of quarks, leptons and neutrinos are of the same origin. We believe that the masses of the quarks and leptons are due to the standard Higgs mechanism. *For neutrino masses a new (or additional) mechanism is needed.* We will discuss here the most popular seesaw mechanism of the generation of small neutrino masses [17]. There are different versions of this mechanism. We will discuss first the mechanism which is based on the Dirac and Majorana mass term.

For illustration let us consider the Dirac and Majorana mass term in the simplest case of one generation. We have

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} m_L \bar{\nu}_L (\nu_L)^c - m_D \bar{\nu}_L \nu_R - \frac{1}{2} m_R \overline{(\nu_R)^c} \nu_R + \text{h.c.}, \quad (38)$$

where $m_{L,R}$ and m_D are real parameters. The mass term (38) can be easily

diagonalized. We have

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} \sum_{i=1,2} m_i \bar{\nu}_i \nu_i, \quad (39)$$

where $\nu_{1,2}$ are Majorana fields with masses $m_{1,2}$ and

$$\nu_L = \cos \theta \nu_{1L} + \sin \theta \nu_{2L}, \quad (\nu_R)^c = -\sin \theta \nu_{1L} + \cos \theta \nu_{2L}. \quad (40)$$

The neutrino masses $m_{1,2}$ and the mixing angle θ are connected with the parameters $m_{L,R}$ and m_R by the following relations

$$m_{1,2} = \frac{1}{2} \left| (m_R + m_L) \mp \sqrt{(m_R - m_L)^2 + 4 m_D^2} \right| \quad (41)$$

and

$$\tan 2\theta = \frac{2m_D}{m_R - m_L}. \quad (42)$$

We will assume now that

1. There is no left-handed Majorana mass term in the Lagrangian, i.e. $m_L = 0$
2. The Dirac mass term is generated by the Standard Higgs mechanism, i.e. m_D is of the order of a mass of a quark or a lepton.
3. A new mechanism generates a right-handed Majorana mass term. This term does not conserve the lepton number. We assume that the lepton number is violated at a scale which is much larger than the electroweak scale, i.e. that $m_R \equiv M_R \gg m_D$.

From (41) and (42) we obtain ⁶

$$m_1 \simeq \frac{m_D^2}{M_R} \ll m_D, \quad m_2 \simeq M_R \gg m_D, \quad \theta \simeq \frac{m_D}{M_R} \ll 1. \quad (43)$$

Thus, in the example we have considered there are two masses in the Majorana mass spectrum: very light (neutrino mass) and very heavy (mass of a new particle). The mixing angle is tiny.

⁶If $m_D \simeq m_t \simeq 170$ GeV and $m_1 \simeq 5 \cdot 10^{-2}$, we find $M_R \simeq 10^{15}$ GeV.

In the case of three families the seesaw matrix has the form

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \quad (44)$$

Here m_D and $M_R = M_R^T$ are 3×3 matrices and $M_R \gg m_D$. The matrix M can be presented in block-diagonal form by the unitary transformation

$$U^T M U = \begin{pmatrix} -m_D M_R^{-1} m_D^T & 0 \\ 0 & M_R \end{pmatrix}. \quad (45)$$

The 3×3 Majorana mass matrix is given by

$$m_\nu = -m_D M_R^{-1} m_D^T. \quad (46)$$

There are many parameters in the matrix m_ν . The large denominator M_R^{-1} ensures, however, the smallness of the neutrino masses with respect to the masses of leptons and quarks. We can make the following conclusions:

1. In the seesaw approach neutrinos with definite masses are Majorana particles.
2. The smallness of neutrino masses is due to a right-handed Majorana mass term which violates the lepton number at a large scale. The suppression factors which provide the smallness of neutrino masses is characterized by the ratio of the electroweak scale and the scale of the violation of the lepton number.
3. Heavy Majorana particles, partners of light Majorana neutrinos, must exist.

We have discussed the seesaw idea in terms of the Dirac and Majorana mass term. The same idea can be realized in another way. Let us assume that there exist heavy Majorana fermions N_i , singlets of the $SU(2) \times U(1)$ group, which have the following $SU(2) \times U(1)$ invariant Yukawa interaction with leptons and standard Higgs bosons

$$\mathcal{L} = \sqrt{2} \sum_{i,l} Y_{il} \bar{L}_{lL} N_{iR} \tilde{\phi} + \text{h.c.} \quad (47)$$

Here Y_{il} are dimensionless constants and

$$L_{lL} = \begin{pmatrix} \nu_{lL} \\ l_L \end{pmatrix} \quad \phi = \begin{pmatrix} \phi^{(+)} \\ \phi^{(0)} \end{pmatrix} \quad (48)$$

are lepton and Higgs doublets and $\tilde{\phi} = i\tau_2\phi^*$ is the conjugated Higgs doublet. We assume that $M_i \gg v$ where M_i is the mass of the Majorana fermion N_i and $v \simeq 246$ GeV is the Higgs vacuum expectation value. It is obvious that the Lagrangian (47) does not conserve L . For the processes with virtual N_i at $Q^2 \ll M_i^2$ the interaction (47) generates a non-renormalizable effective Lagrangian [18]

$$\mathcal{L}_{\text{eff}} = - \sum_{l', l, i} \bar{L}_{l'L} \tilde{\phi} Y_{il'} \frac{1}{M_i} Y_{il} C \tilde{\phi}^T (\bar{L}_{lL})^T + \text{h.c.} \quad (49)$$

If we put

$$\tilde{\phi} = \begin{pmatrix} \frac{v+H}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (50)$$

(H is the Higgs field) the electroweak symmetry will be spontaneously broken and from (49) we obtain the left-handed Majorana mass term

$$\mathcal{L}^{\text{M}} = -\frac{1}{2} \sum_{l'l} \bar{\nu}_{l'L} M_{l'l}^L (\nu_{lL})^c + \text{h.c.}, \quad (51)$$

where

$$M^L = Y^T \frac{v^2}{M} Y \quad (52)$$

is the seesaw mass matrix.⁷ The CP violating decays of heavy Majorana fermions N_i in the early Universe are considered as a possible source of the baryon asymmetry of the Universe (see [19]).

4 On the nature of neutrino oscillations

4.1 Introduction

A lot of debates on the nature of neutrino oscillations can be found in the literature (see recent papers [20]). We will discuss here this problem.

⁷The model we have discussed is usually called the seesaw type I model. The model based on the interaction of lepton pairs and the Higgs pair with heavy scalar triplet bosons is called the seesaw type II model and the model based on the interaction of lepton-Higgs pairs with heavy Majorana triplet fermions is called the seesaw type III model.

From our point of view *the Heisenberg uncertainty relation and the time-energy uncertainty relation are crucial for the phenomenon of neutrino oscillations*. Uncertainty relations in Quantum Theory are based on the inequality

$$\Delta A \Delta B \geq \frac{1}{2} |\langle a | [A, B] | a \rangle|, \quad (53)$$

which can be easily derived from the Cauchy inequality. Here A and B are hermitian operators, $|a\rangle$ is any state and $\Delta A = \sqrt{A^2 - \overline{A}^2}$ is the standard deviation. For example, for operators p and q which satisfy the relation $[p, q] = \frac{1}{i}$ we obtain from (53) the Heisenberg uncertainty relation $\Delta p \Delta q \geq \frac{1}{2}$.

There exist different derivations of the time-energy uncertainty relation

$$\Delta E \Delta t \geq 1. \quad (54)$$

and different interpretations of the quantities which enter into this relation (see, for example, [21]). Mandelstam and Tamm [22] derived the relation (54) from inequality (53) and the evolution equation

$$i \frac{dO(t)}{dt} = [O(t), H] \quad (55)$$

for an operator $O(t)$ in the Heisenberg representation (H is the total Hamiltonian). From (53) and (55) we have

$$\Delta E \Delta O(t) \geq \frac{1}{2} \left| \frac{d}{dt} \overline{O}(t) \right|. \quad (56)$$

For stationary states (56) is identically satisfied. Nontrivial constraints can be obtained only in the case of nonstationary states. In [22] the time-energy uncertainty relation (54) was derived in which ΔE is the uncertainty of the energy of the system and Δt *is the time interval during which the state of the system is significantly changed*.

4.2 Flavor neutrino states

We will consider the neutrino production. Neutrinos are produced in weak decays and reactions. Let us consider (in the lab. system) the decay [23]

$$a \rightarrow b + l^+ + \text{neutrino} \quad (57)$$

where a and b are some hadrons. The sum of the states of the final particles is given by

$$|f\rangle = \sum_i |b\rangle |l^+\rangle |\nu_i\rangle \langle b l^+ \nu_i | S | a \rangle, \quad (58)$$

where $\langle b l^+ \nu_i | S | a \rangle$ is the matrix element of the transition $a \rightarrow b + l^+ + \nu_i$ where ν_i is the neutrino with mass m_i . We assume, as usual, that initial and final particles have definite momenta. Momenta of neutrinos with mass m_i will be denoted by p_i .

Neutrinos ν_i differ only by their masses. If masses of neutrinos are the same, their momenta will be equal. Taking into account that neutrino masses are much smaller than neutrino momenta we have

$$p_i \simeq p + a \frac{\Delta m_{1i}^2}{2E}, \quad (59)$$

where p is the momentum of the lightest neutrino, $E \simeq p$ is the neutrino energy and $|a| \lesssim 1$ is a constant. For the difference of the neutrino momenta we have

$$|p_i - p_k| \lesssim \frac{|\Delta m_{ik}^2|}{2E} = \frac{1}{l_{ik}} \quad (60)$$

For reactor and atmospheric (accelerator) neutrinos we find, respectively,

$$l_{12} \simeq 15 \text{ km}, \quad l_{23} \simeq 200 \text{ km}. \quad (61)$$

For the uncertainty of the neutrino momentum we have

$$(\Delta p)_{QM} \simeq \frac{1}{d}, \quad (62)$$

where d characterizes the quantum-mechanical size of the source.

Because the macroscopic length l_{ik} is much larger than the microscopic quantum-mechanical size of the source we have

$$|p_i - p_k| \ll (\Delta p)_{QM}. \quad (63)$$

Thus, due to the uncertainty relation it is impossible to resolve the momenta of neutrinos with different masses. Because $E_i \simeq p_i(1 + \frac{m_i^2}{2E^2})$ and $\frac{m_i^2}{2E^2} \leq 10^{-13}$, energies of neutrinos with different masses also can not be resolved.

Let us consider the lepton part of the matrix element $\langle b l^+ \nu_i | S | a \rangle$. Taking into account inequality (60), we have

$$U_{li}^* \bar{u}_L(p_i) \gamma_\alpha u(-p_l) \simeq U_{li}^* \bar{u}_L(p) \gamma_\alpha u(-p_l), \quad (64)$$

where p_l is the momentum of l^+ . For the total matrix element we have

$$\langle b \, l^+ \nu_i | S | a \rangle \simeq U_{li}^* \langle b \, l^+ \nu_l | S | a \rangle_{SM}, \quad (65)$$

where $\langle b \, l^+ \nu_l | S | a \rangle_{SM}$ is the Standard Model matrix element of the emission of the flavor neutrino ν_l ⁸ with momentum p and l^+ in the process

$$a \rightarrow b + l^+ + \nu_l. \quad (66)$$

From (58) and (65) we find

$$|f\rangle = |b\rangle |l^+\rangle |\nu_l\rangle \langle b \, l^+ \nu_l | S | a \rangle_{SM}. \quad (67)$$

Here

$$|\nu_l\rangle = \sum_{i=1}^3 U_{li}^* |\nu_i\rangle \quad (l = e, \mu, \tau) \quad (68)$$

is *the state of the flavor neutrino ν_l* . Thus, due to the smallness of the neutrino mass-squared differences and the uncertainty relation it is impossible to say which massive neutrino is emitted in a weak process. This is the reason why *a coherent superposition of states of neutrinos with different masses is produced*. Let us stress that

- Flavor neutrino states do not depend on the production process (for example, ν_e 's produced in μ -decay and in β -decay are the same particles).
- It is natural to assume that flavor states are characterized by the momentum (if there are no special conditions of neutrino production).
- Flavor states are orthogonal and normalized

$$\langle \nu_{l'} | \nu_l \rangle = \delta_{ll'}. \quad (69)$$

⁸By definition the flavor neutrino ν_l is a particle which is emitted in a weak process together with l^+ and the flavor antineutrino $\bar{\nu}_l$ is a particle which is emitted together with l^- .

4.3 Schrödinger evolution of flavor neutrino states

The evolution equation for states in QFT is the Schrödinger equation

$$i \frac{\partial |\Psi(t)\rangle}{\partial t} = H |\Psi(t)\rangle, \quad (70)$$

where H is the total Hamiltonian. The general solution of this equation is given by

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle. \quad (71)$$

If at $t = 0$ in a CC weak process ν_l is produced, for the neutrino state we have at the time t

$$|\nu_l\rangle_t = \sum_i |\nu_i\rangle e^{-iE_i t} U_{li}^*, \quad (72)$$

where $E_i = \sqrt{p^2 + m_i^2}$.

Neutrinos are detected via observation of weak processes. Let us consider the transition

$$\nu_i + N \rightarrow l' + X. \quad (73)$$

For the matrix element we have

$$\langle l' X | S | \nu_i N \rangle \simeq \langle l' X | S | \nu_{l'} N \rangle_{SM} U_{\nu l i}, \quad (74)$$

where $\langle l' X | S | \nu_{l'} N \rangle_{SM}$ is the SM matrix element of the process

$$\nu_{l'} + N \rightarrow l' + X. \quad (75)$$

From (67), (72) and (74) follows that to the chain of processes $a \rightarrow b + l^+ + \nu_l$, $\nu_l \rightarrow \nu_{l'}$, $\nu_{l'} + N \rightarrow l' + X$ corresponds the *factorized product of amplitudes*

$$\langle l' X | S | \nu_{l'} N \rangle_{SM} \left(\sum_i U_{\nu l i} e^{-iE_i t} U_{li}^* \right) \langle b l^+ \nu_l | S | a \rangle_{SM}. \quad (76)$$

Only the amplitude of the transition $\nu_l \rightarrow \nu_{l'}$

$$\mathcal{A}(\nu_l \rightarrow \nu_{l'}) = \sum_i U_{\nu l i} e^{-iE_i t} U_{li}^* \quad (77)$$

depends on the properties of massive neutrinos (mass-squared differences and mixing angles). The matrix elements of the neutrino production and detection do not depend on any characteristics of individual massive neutrinos.

They are given by the Standard Model. Let us stress that the important property of the factorization (76) is based on the Heisenberg uncertainty relation.

For the probability of the transition $\nu_l \rightarrow \nu_{l'}$ we have

$$P(\nu_l \rightarrow \nu_{l'}) = \left| \sum_i U_{l'i} e^{-i(E_i - E_k)t} U_{li}^* \right|^2 = \left| \sum_{i \neq k} U_{l'i} (e^{-i(E_i - E_k)t} - 1) U_{li}^* + \delta_{ll'} \right|^2. \quad (78)$$

From this expression it is obvious that neutrino oscillations can be observed if the condition⁹

$$|E_i - E_k| t \gtrsim 1, \quad (i \neq k) \quad (79)$$

is satisfied. This inequality is the Mandelstam-Tamm time-energy uncertainty relation. According to this relation a change of the flavor neutrino state in time requires energy uncertainty (nonstationary state). The time interval required for a significant change of the flavor neutrino state (oscillations) is given by $t \simeq \frac{1}{|E_i - E_k|}$.

The inequality (79) can be interpreted in another way: in order to resolve a small energy difference $|E_i - E_k| \simeq \frac{|\Delta m_{ik}^2|}{2E}$ we need a macroscopically large time interval $t \gtrsim \frac{1}{|E_i - E_k|}$. This corresponds to another interpretation of the time-energy uncertainty relation (see [24]).

The time t in the equation (70) is a parameter which in our case describes the propagation of the neutrino signal. For the ultrarelativistic neutrino we have $t \simeq L$, where L is the distance between the neutrino source and the detector. Taking into account this relation and the relation $E_i - E_k = \frac{\Delta m_{ki}^2}{2E}$ from (78) we obtain the standard expression (1) for the neutrino transition probability.

4.4 On other approaches to neutrino oscillations

We will now briefly describe other approaches to neutrino oscillations which were considered in the literature. We will start with the following remark. In many papers (see, for example, [25]) the covariant operator $e^{-iP x}$ (P^α is the operator of the total momentum and $x^\alpha = (t, \vec{x})$ is the space-time point) is applied to the mixed flavor neutrino states (68). If we assume that at point

⁹This is a necessary condition for the observation of the oscillations. Another condition: relatively large mixing angles.

$x = 0$ the flavor neutrino ν_l is produced, we have for the neutrino state at the point x in this case

$$|\nu_l\rangle_x = e^{-iPx} |\nu_l\rangle = \sum_i e^{-ip_i x} U_{li}^* |\nu_i\rangle = \sum_{l'} |\nu_{l'}\rangle \left(\sum_i U_{l'i} e^{-ip_i x} U_{li}^* \right) \quad (80)$$

For the probability of the transition $\nu_l \rightarrow \nu_{l'}$ we find the following expression

$$P(\nu_l \rightarrow \nu_{l'}) = \left| \sum_i U_{l'i} e^{-ip_i x} U_{li}^* \right|^2 = \sum_{i \neq k} U_{l'i} (e^{-i(p_i - p_k)x} - 1) U_{li}^* + \delta_{l'l} \quad (81)$$

Let us assume that $\vec{p}_i = p_i \vec{k}$, where \vec{k} is the unit vector. For the phase difference we have

$$(p_i - p_k)x = (E_i - E_k)t - (p_i - p_k)L \simeq \frac{\Delta m_{ki}^2 L}{2E} + (E_i - E_k)(t - L). \quad (82)$$

Taking into account that for the ultrarelativistic neutrinos $t = L$ we obtain from (81) and (82) the standard expression (1) for the neutrino transition probability.¹⁰ Nevertheless the presented "derivation" of the transition probability is wrong. There are two reasons for that:

- The operator e^{-iPx} is the operator of the evolution of *fields*, *but not states*. In fact, from the translational invariance for a field operator $\psi(x)$ we have

$$i \partial_\alpha \psi(x) = [\psi(x), P_\alpha]. \quad (83)$$

The general solution of this equation has the form

$$\psi(x) = e^{iPx} \psi(0) e^{-iPx}. \quad (84)$$

This equation means that e^{-iPx} is the operator of evolution of fields.

- The flavor state $|\nu_l\rangle$, given by equation (68), which describes the mixture of states with definite momenta, can not depend on x . In fact, we have

$$|\nu_i\rangle = c_{-1}^\dagger(p_i) |0\rangle, \quad (85)$$

¹⁰ In the approach based on the Schrödinger equation, the phase difference is equal to $\frac{\Delta m_{ki}^2}{2E} L$ if the flavor state possesses one momentum. We came here to the same result for the phase difference because the neutrino energies in space and time terms are canceled due to the relation $t \simeq L$.

where $|0\rangle$ is the vacuum state and $c_{-1}^\dagger(p_i)$ is the creation operator of a neutrino with momentum p_i , mass m_i and helicity equal to -1. This operator can not depend on x .

The expression (81) for the transition probability, in which neutrino mass states evolve in space and time, can be treated only in the framework of **relativistic quantum mechanics**. In this case the wave function of a flavor neutrino ν_l , produced in a CC process, is the superposition

$$\psi_{\nu_l}(\vec{x}, t) = \sum_i U_{li}^* \psi_i(\vec{x}, t), \quad (86)$$

where

$$\psi_i(\vec{x}, t) = e^{i(\vec{p}_i \vec{x} - E_i t)} u^{(-1)}(p_i) \quad (87)$$

is the solution of the Dirac equation

$$i\gamma^\alpha \partial_\alpha \psi_i(\vec{x}, t) = m_i \psi_i(\vec{x}, t). \quad (88)$$

From (86) and (87) we find that the normalized probability of the transition $\nu_l \rightarrow \nu_{l'}$ is given by the expression (81) in which $p_i x = E_i t - p_i L$ is the change of the phase of the plane wave at the distance L after the time t . From (81), as we have shown before, the standard expression for the transition probability follows.

Let us stress that

- in the approach based on the relativistic quantum mechanics the notion of flavor neutrino states does not appear.
- the "mixed" wave function $\psi_{\nu_l}(\vec{x}, t)$ does not satisfy the Dirac equation:¹¹

$$i\gamma^\alpha \partial_\alpha \psi_{\nu_l}(\vec{x}, t) = \sum_i U_{li}^* m_i \psi_i(\vec{x}, t) \neq m \psi_{\nu_l}(\vec{x}, t) \quad (89)$$

- in order to obtain from the probability (81), which depends on x and t , the standard transition probability we need to assume that

$$L \simeq t. \quad (90)$$

¹¹If any wave function of a particle with spin 1/2 must satisfy the Dirac equation, QM is not the appropriate framework for neutrino oscillations.

We will now briefly discuss the **wave packet approach to the neutrino oscillations** (see [26] and references therein). We will see that this approach provides the equality (90).

Let us take into account the distribution of momenta of the initial neutrinos determined by the uncertainty relation. For the $\nu_l \rightarrow \nu_{l'}$ transition amplitude we have in this case

$$\mathcal{A}(\nu_l \rightarrow \nu_{l'}) = \sum_i U_{l'i} \int e^{i(\vec{p}_i' \vec{x} - E_i' t)} f(\vec{p}_i' - \vec{p}_i) d^3 p' U_{li}^* \quad (91)$$

Here $E_i' = \sqrt{(\vec{p}_i')^2 + m_i^2}$ and the function $f(\vec{p}_i' - \vec{p}_i)$ has a sharp maximum at the point $\vec{p}_i' = \vec{p}_i$. We assume that $|\vec{p}_i' - \vec{p}_i| \ll p_i$.

Expanding E_i' at the point $\vec{p}_i' = \vec{p}_i$ we have

$$E_i' \simeq E_i + (\vec{p}_i' - \vec{p}_i) \cdot \vec{v}_i, \quad (92)$$

where $E_i = \sqrt{p_i^2 + m_i^2}$ and

$$\vec{v}_i = \frac{\vec{p}_i}{E_i} \quad (93)$$

Taking into account (92) we find

$$\int e^{i(\vec{p}_i' \vec{x} - E_i' t)} f(\vec{p}_i' - \vec{p}_i) d^3 p' = e^{-i(\vec{p}_i \vec{x} - E_i t)} g(\vec{x} - \vec{v}_i t), \quad (94)$$

where the amplitude $g(\vec{x} - \vec{v}_i t)$ is given by the expression

$$g(\vec{x} - \vec{v}_i t) = \int e^{i\vec{q} \cdot (\vec{x} - \vec{v}_i t)} f(\vec{q}) d^3 q. \quad (95)$$

Notice that the wave packet transition amplitude differs from the amplitude in the plane wave approximation by the additional factor g . Because of the relativistic relation between momentum and energy this factor depends on the combination $\vec{x} - \vec{v}_i t$.

Usually it is assumed that the function $f(\vec{q})$ has the Gaussian form

$$f(\vec{q}) = N e^{-\frac{q^2}{4\sigma_p^2}}, \quad (96)$$

where σ_p is the width of the wave packet in the momentum space. From (95) and (96) we find

$$g(\vec{x} - \vec{v}_i t) = N \left(\frac{\pi}{\sigma_x^2} \right)^{3/2} e^{-\frac{(\vec{x} - \vec{v}_i t)^2}{4\sigma_x^2}}, \quad (97)$$

where $\sigma_x = \frac{1}{2\sigma_p}$ characterizes the spacial width of the wave packet.

In the wave packet approach the probability of the transition $\nu_l \rightarrow \nu_{l'}$ is determined as a quantity obtained by integration over time (assuming that in neutrino oscillation experiments time is not measured)

$$P(\nu_l \rightarrow \nu_{l'}) = \int_{-\infty}^{+\infty} |\mathcal{A}(\nu_l \rightarrow \nu_{l'})|^2 dt \quad (98)$$

From (94) and (98) we find the following expression for the normalized transition probability

$$P(\nu_l \rightarrow \nu_{l'}) = \sum_{i,k} U_{l'i} U_{l'k}^* e^{i(p_i - p_k)x} U_{li}^* U_{lk} e^{A_{ik}}, \quad (99)$$

where

$$A_{ik} = -i(E_i - E_k)x - \frac{1}{2\sigma_x^2} \left(\frac{\Delta m_{ik}^2}{4E^2} \right)^2 x^2 - \frac{1}{2} \sigma_x^2 \xi^2 \left(\frac{\Delta m_{ik}^2}{2E} \right)^2. \quad (100)$$

Here $E_i = E + \xi \frac{m_i^2}{2E}$ and ξ is a constant of the order of one.

The factor $e^{A_{ik}}$ is the result of the integration over t . From the first term of the expression for A_{ik} it is evident that the Gaussian amplitude $g(\vec{x} - \vec{v}_i t)$ (after the integration over t) provides the equality $t = x$. For usual neutrino oscillation experiments the second and the third terms of the expression for A_{ik} are very small. In fact, let us introduce the coherence and oscillation lengths¹²

$$L_{\text{coh}}^{ik} = \frac{4\sqrt{2}\sigma_x E^2}{|\Delta m_{ik}^2|}, \quad L_{\text{osc}}^{ik} = 4\pi \frac{E}{|\Delta m_{ik}^2|}. \quad (101)$$

The expression for the transition probability takes the form

$$\mathcal{P}(\nu_l \rightarrow \nu_{l'}) = \sum_{i,k} U_{l'i} U_{l'k}^* e^{i\frac{\Delta m_{ik}^2}{2E} L} U_{li}^* U_{lk} e^{-\left(\frac{L}{L_{\text{coh}}^{ik}}\right)^2} e^{-2\pi^2 \xi^2 \left(\frac{\sigma_x}{L_{\text{osc}}^{ik}}\right)^2}, \quad (102)$$

where $x = L$ is the distance between neutrino source and neutrino detector.

We have

$$L_{\text{coh}}^{ik} = \frac{\sqrt{2}}{\pi} \sigma_x E L_{\text{osc}}^{ik}. \quad (103)$$

¹²We have $|v_i - v_k| L_{\text{coh}}^{ik} \simeq \frac{|\Delta m_{ik}^2|}{2E^2} \sim \sigma_x$. Thus, the coherence length characterizes such a distance between neutrino source and detector at which the distance between ν_i and ν_k becomes comparable to the size of the wave packet.

From this expression It follows that the coherence length is much larger than the oscillation length. Thus, for neutrino oscillation experiments with $L \simeq L_{\text{osc}}^{ik}$ the term $e^{-\left(\frac{L}{L_{\text{coh}}^{ik}}\right)^2}$ is practically equal to one.

Further we have¹³

$$L_{\text{osc}}^{ik} \gg \sigma_x. \quad (104)$$

Thus, the term $e^{-2\pi^2\xi^2\left(\frac{\sigma_x}{L_{\text{osc}}^{ik}}\right)^2}$ is also practically equal to one.

We will finish this part with the following remarks:

- Integration over time in the wave packet approach assumes that the time interval t between neutrino production and detection is not measured in neutrino oscillation experiments. This is correct in the case of the atmospheric and reactor neutrino experiments because the time of neutrino production is not known in such experiments. However, in the case of the accelerator neutrino experiments (K2K, MINOS, T2K) neutrinos are produced in spills and the time of neutrino production is known. In these experiments the time of neutrino production is measured and the time interval t is known. For example, in the K2K experiment [4] the measurement of $t = t_{SK} - t_{KEK}$, where t_{SK} is the time of detection of neutrinos in the Super-Kamiokande detector and t_{KEK} is the time of the production of neutrinos at KEK, allowed to show that

$$-0.2 \leq \left|t - \frac{L}{c}\right| \leq 1.3 \text{ } \mu\text{s}. \quad (105)$$

- The wave packet approach assures the equality $t = L$ and the standard oscillation phase in the transition probability. Two additional exponential factors are very close to one for usual neutrino oscillation experiments. The effect of the decoherence term could be important only for large cosmological distances.

In many papers (see [20]) neutrinos, propagating about 100 km (reactor ν 's) or about 1000 km (atmospheric and accelerator ν 's), are considered as **virtual neutrinos** in a Feynman diagram-like picture with the neutrino production process at one vertex and the neutrino absorption process in another vertex. This approach gives the wave packet picture of neutrino

¹³As we discussed before, because of this inequality coherent flavor neutrino states are produced.

oscillations with a transition probability which (before integration over t) depends on x and t .

In the standard S matrix approach, which is based on the local quantum field theory, the transition amplitude is given by

$$\langle f|S|i\rangle = \langle f|T(e^{-i\int \mathcal{H}_I(x) d^4x})|i\rangle. \quad (106)$$

where $\mathcal{H}_I(x)$ is the interaction Hamiltonian. Let us stress that

- in all orders of the perturbation theory of the matrix element (106) integration over *the same (in our case weak) interaction region* is performed and virtual particles belong to the same region. In the "virtual neutrino approach to neutrino oscillations" there are two interaction regions (production and detection) separated by a large macroscopic distance.
- in the standard S matrix approach initial and final states are states of free particles considered at *the same time* (correspondingly at $t \rightarrow -\infty$ and at $t \rightarrow +\infty$). In the "virtual neutrino approach" initial and final states are states of particles at fixed space-time points separated by macroscopic distance and time.

This "virtual neutrino approach" can be considered as a model based on the combination of field theory and relativistic quantum mechanics. From our point of view, the applicability of this approach to neutrino oscillations requires experimental tests.

5 Conclusions

We have discussed different approaches to neutrino oscillations. The QFT approach is based on the assumption that states of flavor neutrinos ν_l are mixed coherent states $|\nu_l\rangle = \sum_i U_{li}^* |\nu_i\rangle$. The evolution of flavor neutrino states in time is determined by the Schrödinger equation for quantum states. The QFT approach is based on the same general principles as the approach to $B^0 \rightleftharpoons \bar{B}^0$ etc. oscillations studied in detail at B-factories and other facilities. The important characteristic feature of this approach is the Mandelstam-Tamm time-energy uncertainty relation.

Other approaches are based on the assumption that in weak processes mixed coherent superpositions of plane waves or wave packets, describing

neutrinos with different masses, are produced and detected. The evolution of mixed neutrino wave functions in space and time is determined by the Dirac equation.

Different approaches to neutrino oscillations lead to the same expression for the neutrino transition probability $\mathcal{P}(\nu_l \rightarrow \nu_{l'})$ in the standard neutrino oscillation experiments. In order to distinguish different approaches special neutrino oscillation experiments are necessary. Such experiments could be the recently discussed Mössbauer neutrino experiments [27, 28].

As an example let us consider the recoilless Mössbauer transition

$${}^3\text{H} \rightarrow {}^3\text{He} + \bar{\nu}_e, \quad \bar{\nu}_e + {}^3\text{He} \rightarrow {}^3\text{H}. \quad (107)$$

in which a $\bar{\nu}_e$ with energy $\simeq 18.6$ keV is produced and absorbed.

It was estimated in [27] that the uncertainty of the energy of the antineutrino in the Mössbauer transition (107) is of the order

$$(\Delta E)_{\text{M}} \simeq 8.4 \cdot 10^{-12} \text{ eV}. \quad (108)$$

Let us compare $(\Delta E)_{\text{M}}$ given by (108) with the quantity $\frac{\Delta m_A^2}{2E}$ which could govern neutrino oscillations in (107). We have

$$\frac{\Delta m_A^2}{2E} \simeq 0.6 \cdot 10^{-7} \text{ eV} \quad (109)$$

Thus, we have

$$(\Delta E)_{\text{M}} \ll \frac{\Delta m_A^2}{2E}. \quad (110)$$

This means that neutrino oscillations with the oscillation length given by $L_{\text{osc}}^{\text{A}} = 4\pi \frac{E}{\Delta m_A^2}$ can not be observed in the Mössbauer neutrino experiment if the QFT approach is valid [29]. This statement is in agreement with the time-energy uncertainty relation: the uncertainty of the energy in the Mössbauer transition is too small to fulfill the Mandelstam-Tamm time-energy uncertainty relation (54) with $t \simeq L_{\text{osc}}^{\text{A}}$.

On the other side, if the space-time picture of neutrino oscillations is valid, neutrino oscillations with the oscillation length $L_{\text{osc}}^{\text{A}}$ will be observed in the Mössbauer neutrino experiment [30]. In fact, for the oscillation phase we have in this case

$$(E_3 - E_2) - (p_3 - p_2) \simeq \frac{\Delta m_A^2}{2E}. \quad (111)$$

In the space-time approach a significant change of the neutrino state at the distance L_{osc}^A does not require a corresponding energy uncertainty. In other words, neutrino oscillations in the space-time approach do not necessarily follow the Mandelstam-Tamm uncertainty relation.

Neutrino oscillations (like $B^0 \rightleftharpoons \bar{B}^0$ etc. oscillations) are an extremely important quantum phenomenon. Because of the interference nature of neutrino oscillations their investigation allows to determine tiny neutrino mass-squared differences which are not reachable in other experiments. The theory of neutrino oscillations is grounded on basic conceptions. The study of neutrino oscillations in a Mössbauer neutrino experiment with practically monoenergetic neutrinos would allow us to answer such fundamental questions of Quantum Theory as the problem of the existence of mixed coherent flavor states, the problem of the evolution of the quantum states (in time or in space and time), the problem of the universal applicability of the time-energy uncertainty relation and others.

I would like to express my deep gratitude to Walter Potzel for numerous discussions. This work has been supported by funds of the DFG (Transregio 27: Neutrinos and Beyond), the Munich Cluster of Excellence (Origin and Structure of the Universe), and the Maier-Leibnitz-Laboratorium (Garching).

References

- [1] B. T. Cleveland *et al.* (Homestake), *Astrophys. J.* **496**, 505 (1998); J. N. Abdurashitov *et al.* (SAGE), *Phys. Rev.* **C80**, 015807 (2009), arXiv:0901.2200; W. Hampel *et al.* (GALLEX), *Phys. Lett.* **B447**, 127 (1999); M. Altmann *et al.* (GNO), *Phys. Lett.* **B616**, 174 (2005), hep-ex/0504037; J. P. Cravens *et al.* (Super-Kamiokande), *Phys. Rev.* **D78**, 032002 (2008), arXiv:0803.4312; B. Aharmim *et al.* (SNO), *Phys.Rev.***C81**, 055504 (2010), arXiv:00910.2984v2.
- [2] R. Wendell *et al.*, (Super-Kamiokande) *Phys. Rev.* **D81**, 09200 (2010); arXiv:1002.3471.
- [3] Gando *et al.* (KamLAND), arXiv:1009.4771v2.
- [4] M.H.Ahn *et al.* (K2K), *Phys. Rev.* **D74**, 072003 (2006), hep-ex/0606032.
- [5] A.Habig *et al.*(MINOS), *Mod. Phys. Lett.* **A25**, 1219 (2010), arXiv:1004.2647.

- [6] A.Aguilar *et al.* (LSND), Phys. Rev. **D64**, 112007 (2001).
- [7] A.A. Aguilar-Arevalo *et al.* (MiniBooNE), Phys. Rev. Lett. **105**, 181801 (2010), arXiv:1007.1150.
- [8] A.A. Aguilar-Arevalo *et al.* (MiniBooNE), Phys. Rev. Lett.**98**, 231801 (2007), arXiv:0704.1500.
- [9] S.M. Bilenky, C. Giunti, and W. Grimus, Prog. Part. Nucl. Phys. **43**, 1 (1999).
- [10] B. Pontecorvo, J. Exptl. Theoret. Phys. **33**, 549 (1957). [Sov. Phys. JETP **6**, 429 (1958)]; J. Exptl. Theoret. Phys. **34**, 247 (1958) [Sov. Phys. JETP **7**, 172 (1958)].
- [11] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
- [12] M. Apollonio *et al.* (CHOOZ), Eur. Phys. J. **C27**, 331 (2003), arXiv: hep-ex/0301017.
- [13] Ch. Kraus *et al.* (Mainz), Eur.Phys.J. **C40**, 447 (2005), arXiv:hep-ex/0412056.
- [14] V.M. Lobashev *et al.* (Troitsk), Prog. Part. Nucl. Phys. **48**, 123 (2002).
- [15] V.N. Gribov and B. Pontecorvo, Phys. Lett. **B28**, 493 (1969).
- [16] A.D. Dolgov and A Yu. Smirnov arXiv: hep-ph/0501066. A.S. Barabash, Found. Phys.**40**, 703 (2010), arXiv:0908.3795.
- [17] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, p. 315, edited by F. van Nieuwenhuizen and D. Freedman, North Holland, Amsterdam, 1979; T. Yanagida, Proc. of the *Workshop on Unified Theory and the Baryon Number of the Universe*, KEK, Japan, 1979; R.N. Mohapatra and G. Senjanović, Phys. Rev. Lett. **44**, 912 (1980); P. Minkowski, Phys. Lett. **B67**, 421 (1977).
- [18] S. Weinberg, Phys. Rev. Lett. **43**, 1571 (1979).
- [19] S. Davidson, E. Nardi, Y. Nir, Phys. Rept. **466**, 105 (2008), arXiv:0802.2962.

- [20] M. Beuthe, Phys. Rep. **375**, 105 (2003), arXiv:hep-ph/0109119. C. Giunti, AIP Conf.Proc.1026:3-19,(2008), arXiv:0801.0653. E. Akhmedov, J. Kopp, JHEP**04** 008(2010), arXiv:1001.4815v2; D. V. Naumov and V.A. Naumov, J. Phys. G: Nucl. Part. Phys. **37**, 105014 (2010), arXiv:1008.0306v2; J. Wu, J. A. Hutasoit, D. Boyanovsky, R. Holman, arXiv:1002.2649.
- [21] P. Busch, arXiv:quant-ph/0105049v3.
- [22] L. Mandelstam and I. Tamm, J. Phys. (USSR) **9**, 249 (1945).
- [23] S. Bilenky and C. Giunti, Int. J. Mod. Phys.**A16**, 3931 (2001), arXiv: hep-ph/0102320.
- [24] V. Fock and N. Krylov, Journal of Physics, **v. 11 N.2**, 112 (1947).
- [25] J.M. Levy, hep-ph/0004221.
- [26] C. Giunti, Found. Phys. Lett. **17**, 103 (2004), arXiv: hep-ph/0302026.
- [27] R.S. Raghavan, arXiv: hep-ph/0601079v3.
- [28] W. Potzel, Phys. Scr. **T127**, 85 (2006); W. Potzel , Acta Phys. Polon. **840**,3033 (2009); J. Phys. Conf. Ser. **136**, 022010 (2008), arXiv: 0810.2170.
- [29] S.M. Bilenky, F. von Feilitzsch, and W. Potzel, J. Phys. **G34**, 987 (2007), arXiv: hep-ph/0611285; S.M. Bilenky, F. von Feilitzsch, and W. Potzel, J. Phys. **G35**, 095003 (2008), arXiv: 0803.0527.
- [30] E.Kh. Akhmedov, J. Kopp, and M. Lindner, JHEP **0805**, 005 (2008), arXiv: 0802.2513.